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# Nonlinear Gauge Transformations and Exact Solutions of the Doebner-Goldin Equation

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**Abstract**

Invariants of nonlinear gauge transformations of a family of nonlinear Schrödinger equations proposed by Doebner and Goldin are used to characterize the behaviour of exact solutions of these equations.

## 1 Introduction

In this paper we shall exhibit solutions for members of the family of nonlinear Schrödinger equations derived by Doebner and Goldin [1, 2]. Originally these equations were derived in a quantum mechanical context and proposed as possible nonlinear quantum evolution equations. There are various contributions in this volume dealing with certain quantum mechanical aspects of this family. E.g. Goldin [3] reviews a derivation of the equation and discusses gauge transformations of the family, Hennig [4] gives a geometric derivation, Lücke [5] discusses quantum mechanical observables in nonlinear evolution equations and, Malomed [6] et.al. discuss the stability of plain wave solutions, and Mizrahi and Dodonov [7] consider an equation generalizing the original equation in [1] and Gaussian solutions thereof.

Owing to the presence of many parameters the above mentioned family is very large and certain members may be of interest in other fields of physics as well, e.g. [8]–[14]. Consequently we shall not confine our interest to only certain parameter values (e.g. "small nonlinearity") but attempt to construct solutions for all values.

The family of nonlinear Schrödinger equations [2] to be investigated is parameterized by the classification parameter $D$ of the unitarily inequivalent group representations and five real

parameters $D'c_1, \ldots, D'c_5$:

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V(\vec{x})\right)\psi + i\frac{\hbar D}{2}\frac{\Delta\rho}{\rho}\psi + \hbar D'\left(\sum_{j=1}^{5} c_j R_j[\psi]\right)\psi\,. \tag{1}$$

Here $D'$ also has the dimensions of a diffusion coefficient (so that the $c_j$ are dimensionless), and the nonlinear functionals $R_j$ are complex homogeneous of degree zero, defined by:

$$R_1[\psi] := \frac{\vec{\nabla}\cdot\vec{J}}{\rho}\,, \qquad R_2[\psi] := \frac{\Delta\rho}{\rho}\,, \qquad R_3[\psi] := \frac{\vec{J}^2}{\rho^2}\,,$$
$$R_4[\psi] := \frac{\vec{J}\cdot\vec{\nabla}\rho}{\rho^2}\,, \qquad R_5[\psi] := \frac{(\vec{\nabla}\rho)^2}{\rho^2}\,,$$

where $\rho := \bar{\psi}\psi$ and $\vec{J} := \mathrm{Im}\,(\bar{\psi}\vec{\nabla}\psi) = (m/\hbar)\vec{j}$. Our analysis is facilitated if we rewrite the family (1) wholly in terms of densities and currents. Using the expansion of the Laplacian $\Delta\psi = (iR_1[\psi] + (1/2)R_2[\psi] - R_3[\psi] - (1/4)R_5[\psi])\psi$, equation (1) is of the general form

$$\mathcal{S}_{(\nu,\mu)}(\psi) := i\partial_t\psi - i\sum_{j=1}^{2}\nu_j R_j[\psi]\psi - \sum_{j=1}^{5}\mu_j R_j[\psi]\psi - \mu_0 V\psi = 0 \tag{2}$$

where $\nu = (\nu_1, \nu_2)$ and $\mu = (\mu_0, \ldots, \mu_5)$ are real parameters and in particular $\nu_1 \neq 0$.

Writing $\psi$ in the form

$$\psi = \exp(\theta_1 + i\theta_2) \tag{3}$$

and $\theta = (\theta_1, \theta_2)$ one finds the following equations:

$$\mathcal{AP}_{(\nu,\mu)}(\theta)_1 \;:=\; \partial_t\theta_1 - 2\nu_2\Delta\theta_1 - \nu_1\Delta\theta_2 - 4\nu_2(\vec{\nabla}\theta_1)^2 - 2\nu_1\vec{\nabla}\theta_1\cdot\vec{\nabla}\theta_2 = 0 \tag{4}$$

$$\begin{aligned}\mathcal{AP}_{(\nu,\mu)}(\theta)_2 \;:=\; \partial_t\theta_2 &+ 2\mu_2\Delta\theta_1 + \mu_1\Delta\theta_2 + 4(\mu_2+\mu_5)(\vec{\nabla}\theta_1)^2 \\ &+ 2(\mu_1+\mu_4)\vec{\nabla}\theta_1\cdot\vec{\nabla}\theta_2 + \mu_3(\vec{\nabla}\theta_2)^2 + \mu_0 V = 0\,.\end{aligned} \tag{5}$$

We shall refer to (4) as the amplitude and to (5) as the phase equation. Due to the ambiguity of the phase function $\theta_2$ in (3) the nonlinear Schrödinger equation (2) and the amplitude and phase equation are *not* fully equivalent. However, any solution of the amplitude and phase equation yields via (3) a solution of the nonlinear Schrödinger equation (2), i.e. $\mathcal{AP}_{(\nu,\mu)}(\theta) = 0 \Rightarrow \mathcal{S}_{(\nu,\mu)}(\psi) = 0$.

## 2 Nonlinear gauge transformations

Before we calculate solutions of the set (2) of nonlinear partial differential equations (PDEs) (2), let us look for invertible linear transformations of the real functions $\theta_1$, $\theta_2$ leaving the structure of the amplitude and phase equations (4,5) invariant, i.e. we are looking for a matrix $A \in GL(2)$, such that

$$\theta'(\vec{x},t) := \begin{pmatrix}\theta_1'(\vec{x},t)\\ \theta_2'(\vec{x},t)\end{pmatrix} := \begin{pmatrix}A_{11}\theta_1(\vec{x},t) + A_{12}\theta_2(\vec{x},t)\\ A_{21}\theta_1(\vec{x},t) + A_{22}\theta_2(\vec{x},t)\end{pmatrix} = A\,\theta(\vec{x},t) \tag{6}$$

solve some PDE of the set with parameters $(\nu', \mu')$, if $\theta$ was a solution of (4,5). A short calculation shows that the matrix has to be restricted to

$$A(\Lambda,\gamma) = \begin{pmatrix} 1 & 0 \\ \gamma & \Lambda \end{pmatrix}, \tag{7}$$

where $\Lambda \neq 0$, hence to a representation of the one dimensional affine group $\mathit{Aff}(1)$. The change of parameters according to this transformation is given by

$$(\nu,\mu) \mapsto (\nu',\mu') = A(\Lambda,\gamma).(\nu,\mu)\,, \tag{8}$$

where

$$\begin{gathered} \nu_1' = \frac{\nu_1}{\Lambda}\,, \quad \nu_2' = -\frac{\gamma}{2\Lambda}\nu_1 + \nu_2\,, \\ \mu_1' = -\frac{\gamma}{\Lambda}\nu_1 + \mu_1\,, \quad \mu_2' = \frac{\gamma^2}{2\Lambda}\nu_1 - \gamma\nu_2 - \frac{\gamma}{2}\mu_1 + \Lambda\mu_2\,, \quad \mu_3' = \frac{\mu_3}{\Lambda}\,, \\ \mu_4' = -\frac{\gamma}{\Lambda}\mu_3 + \mu_4\,, \quad \mu_5' = \frac{\gamma^2}{4\Lambda}\mu_3 - \frac{\gamma}{2}\mu_4 + \Lambda\mu_5\,, \quad \mu_0' = \Lambda\mu_0\,. \end{gathered} \tag{9}$$

The corresponding transformation of the wave function $\psi$ is the one used to linearize a certain subfamily of (1) in [15] and partly in [16]:

$$N_{A(\Lambda,\gamma)}(\psi) = \psi^{\frac{1}{2}(1+\Lambda+i\gamma)}\bar{\psi}^{\frac{1}{2}(1-\Lambda+i\gamma)}\,. \tag{10}$$

Since these transformations leave the probability density $\rho_\psi(\vec{x},t) = \psi(\vec{x},t)\bar{\psi}(\vec{x},t)$ invariant, they were called *nonlinear gauge transformations* [17] (see also [3] in this volume). Starting with a solution of the amplitude and phase equation we can use transformations $A \in \mathit{Aff}(1)$ to construct solutions of the nonlinear Schrödinger equations of the type (2) by exploiting the implications represented in the following diagram:

$$\begin{array}{ccc} \mathcal{AP}_{(\nu,\mu)}(\theta) = 0 & \overset{(6)}{\Longleftrightarrow} & \mathcal{AP}_{A.(\nu,\mu)}(A\,\theta) = 0 \\ \Downarrow & \psi = \exp(\theta_1 + i\theta_2) & \Downarrow \\ \mathcal{S}_{(\nu,\mu)}(\psi) = 0 & & \mathcal{S}_{A.(\nu,\mu)}(N_A(\psi)) = 0 \end{array} \tag{11}$$

The change of parameters (9) amounts to the action of the affine group on the eight-parameter space, so it is worthwhile to look for a parameterization of (2) that consists of six parameters invariant under the action of $\mathit{Aff}(1)$ and two group parameters. Six functionally independent invariants have been proposed in [17],

$$\begin{gathered} \iota_1 = \nu_1\mu_2 - \nu_2\mu_1\,, \quad \iota_2 = \mu_1 - 2\nu_2\,, \quad \iota_3 = 1 + \mu_3/\nu_1\,, \quad \iota_4 = \mu_4 - \mu_1\mu_3/\nu_1\,, \\ \iota_5 = \nu_1(\mu_2 + 2\mu_5) - \nu_2(\mu_1 + 2\mu_4) + 2\nu_2^2\mu_3/\nu_1\,, \quad \iota_0 = \nu_1\mu_0\,, \end{gathered} \tag{12}$$

and as group parameters we might choose $\nu_1 \neq 0$ and $\mu_1$. Re-expressing the other parameters in terms of the new invariants we get

$$\begin{aligned} \nu_2 = \frac{1}{2}(\mu_1 - \iota_2)\,, \quad \mu_2 = \frac{1}{2}\nu_1^{-1}(2\iota_1 - \iota_2\mu_1 + \mu_1^2)\,, \\ \mu_3 = (\iota_3 - 1)\nu_1\,, \quad \mu_4 = \iota_4 - \mu_1 + \iota_3\mu_1\,, \\ \mu_5 = \frac{1}{2}\nu_1^{-1}\left(\iota_5 - \iota_1 + \iota_4(\mu_1 - \iota_2) + \frac{1}{2}(\mu_1^2 - \iota_2^2)(\iota_3 - 1)\right)\,, \quad \mu_0 = \nu_1^{-1}\iota_0\,. \end{aligned} \tag{13}$$

For the gauge class containing the linear Schrödinger equation (1) ($D = D' = 0$) the invariants read

$$\iota_0 = -\frac{1}{2m}\,, \qquad \iota_1 = \frac{\hbar^2}{8m^2}\,, \qquad \iota_j = 0 \quad j = 2, \ldots, 5\,, \tag{14}$$

such that linear Schrödinger equations are always represented by invariants of the type $\iota_0 < 0, \iota_1 > 0, \iota_2 = \ldots = \iota_5 = 0$.

For the calculation of solutions of (2) we might limit ourselves to a particular *gauge*, i.e. to a particular choice of the group parameters $\nu_1, \mu_1$. An appropriate choice is $\nu_1 = 1$ and $\mu_1 = 0$, and it is sufficient to consider the amplitude and phase equation in this particular “gauge”

$$\partial_t\theta_1 = -\iota_2\Delta\theta_1 + \Delta\theta_2 - 2\iota_2(\vec{\nabla}\theta_1)^2 + 2\vec{\nabla}\theta_1 \cdot \vec{\nabla}\theta_2\,, \tag{15}$$

$$\begin{aligned} \partial_t\theta_2 = & -2\iota_1\Delta\theta_1 - \left(2\iota_5 + 2\iota_1 - 2\iota_4\iota_2 - \iota_2^2(\iota_3 - 1)\right)(\vec{\nabla}\theta_1)^2 \\ & -2\iota_4\vec{\nabla}\theta_1 \cdot \vec{\nabla}\theta_2 - (\iota_3 - 1)(\vec{\nabla}\theta_2)^2 - \iota_0 V\,. \end{aligned} \tag{16}$$

We will now exhibit various solutions for these equations along with the fact that the “gauge invariant” parameters may be used to characterize their different behaviour.

# 3 Stationary solutions

Stationarity is defined as usual by $\partial_t\rho = 0$ which is equivalent to $\partial_t\theta_1 = 0$. Hence, for stationary solutions the amplitude equation (15) reduces to

$$\Delta(\iota_2\theta_1 - \theta_2) + \vec{\nabla}\theta_1 \cdot \vec{\nabla}(\iota_2\theta_1 - \theta_2) = 0. \tag{17}$$

For this equation we have two obvious solutions leading to the following cases (and subsequent sub-cases).

## 3.1 Plane waves

Plane wave solutions are obtained if $V \equiv 0$ in which case setting $\theta_1(\vec{x}, t) = const$ in (17) yields $\theta_2(\vec{x}, t) = \vec{k} \cdot \vec{x} + k(t)$ from the amplitude equation such that the resulting phase equation (16) gives a dispersion relation:

$$\partial_t\theta_2(\vec{x}, t) = (1 - \iota_3)k^2 = (1 + \frac{2mDc_3}{\hbar})k^2 =: \omega(\vec{k}) \tag{18}$$

and the solutions thus found are

$$\psi(\vec{x},t)=\psi_0\exp\left(i(\vec{k}\cdot\vec{x}-\omega(\vec{k}))\right)\,,\qquad \psi_0=const. \tag{19}$$

## 3.2 Nontrivial stationary solutions

Neglecting constant phase factors the Ansatz

$$\theta_2(\vec{x},t)=\iota_2\theta_1(\vec{x})-\omega t \tag{20}$$

also solves (17) and leads to the following phase equation

$$2\iota_1\Delta\theta_1+2(\iota_1+\iota_5)(\vec{\nabla}\theta_1)^2+\iota_0 V=\omega\,. \tag{21}$$

### 3.2.1 Stationary solutions from linear Schrödinger equations

If $\iota_1\neq 0\neq\iota_1+\iota_5$ and $\theta_1$ is a solution of (21) then the function

$$\varphi(\vec{x}):=\exp\left(\frac{\iota_1+\iota_5}{\iota_1}\theta_1(\vec{x})\right) \tag{22}$$

satisfies

$$\frac{2\iota_1^2}{\iota_0(\iota_1+\iota_5)}\Delta\varphi+V\varphi=\frac{\omega}{\iota_0}\varphi. \tag{23}$$

From the solutions $\varphi$ of this equation one obtains the following stationary solutions of the nonlinear equation (1)

$$\psi(\vec{x},t)=(\varphi(\vec{x}))^{\frac{\iota_1}{\iota_1+\iota_5}}\exp\left(i\frac{\iota_1\iota_2}{\iota_1+\iota_5}\ln[\varphi(\vec{x})]^2-i\omega t\right). \tag{24}$$

As an example we calculate the ground state of the harmonic oscillator in one dimension $V(x)=\frac{\kappa}{2}x^2$, for $\iota_0(\iota_1+\iota_5)<0$

$$\psi_0(x,t)=\exp\left(-\frac{1}{4}\sqrt{\frac{-\kappa\iota_0}{\iota_1+\iota_5}}\,x^2+i\left(-\frac{\iota_2}{2}\sqrt{\frac{-\kappa\iota_0}{\iota_1+\iota_5}}\,x^2-|\iota_1|\sqrt{\frac{-\kappa\iota_0}{\iota_1+\iota_5}}\,t\right)\right)\,. \tag{25}$$

### 3.2.2 Other stationary solutions

If $\iota_1=0\neq\iota_5$ the phase equation reduces to

$$2\iota_5\left(\vec{\nabla}\theta_1\right)^2+\iota_0 V=\omega\,, \tag{26}$$

which obviously can only be solved for potentials bounded from above ($\iota_0>0$) or below ($\iota_0<0$). For such bounded potential separable in Cartesian coordinates $V(\vec{x},t)=\sum_{j=1}^n V_j(x_j,t)$ we get

$$\theta_1(\vec{x},t)=\sum_{j=1}^n\int^{x_j}\sqrt{\frac{C_j-\iota_0V_j(\xi,t)}{2\iota_5}}d\xi+C\,,\quad \sum_{j=1}^n C_j=\omega\,,$$

or in particular for the free case ($V \equiv 0$) $\theta_1(\vec{x}, t) = \vec{k} \cdot \vec{x} + const$ and $\omega = 2\iota_5 k^2$, hence the solution

$$\psi(\vec{x}, t) = \psi_0 \exp\left((1 + i\iota_2)\vec{k} \cdot \vec{x} - i2\iota_5 k^2 t\right) , \qquad \psi_0 = const. \tag{27}$$

which is not square integrable.

If on the other hand $\iota_1 + \iota_5 = 0 \neq \iota_1$ the phase equation reduces to

$$2\iota_1 \Delta\theta_1 = \omega - V , \tag{28}$$

i.e. a Poisson equation for $\theta_1$, which may be solved by a Poisson integral, if $V(\vec{x}, t)$ is suitably decreasing at infinity

$$\theta_1(\vec{x}, t) = \eta x^2 + \vec{k} \cdot \vec{x} + C - (2\iota_1)^{-1} \int_{I\!R^n} G_n(\vec{x}, \vec{y}) V(\vec{y}, t) d^n y , \qquad \omega = 4\iota_1 \eta ,$$

where $G_n$ is Green's function for the Poisson equation in $n$ dimensions. Thus in the free case the solution $\theta_1(\vec{x}, t) = \vec{k} \cdot \vec{x} + V + const$ leads to the wave function

$$\psi(\vec{x}, t) = \psi_0 \exp\left(\eta x^2 + \vec{k} \cdot \vec{x} + 2i\iota_2(\eta x^2 + \vec{k} \cdot \vec{x}) - 4i\iota_1 \eta t\right) , \qquad \psi_0 = const \tag{29}$$

which is square integrable only if $\eta < 0$.

Finally, if $\iota_1 = 0 = \iota_5$ then the free phase equation is satisfied by arbitrary $\theta_1$ with $\omega = 0$ and we have the following stationary solution

$$\psi(\vec{x}, t) = \psi_0 \exp\left((1 + 2i\iota_2)\theta_1(\vec{x})\right) , \qquad \psi_0 = const. \tag{30}$$

# 4 Gaussian non-stationary solutions

## 4.1 Gaussian wave Ansatz

In the following we shall consider either the case where a harmonic potential $V(\vec{x}) = \frac{\kappa}{2} x^2$ is present or the free case which is thus described by $\kappa = 0$ [18, 19], for a special case [20]. Since these cases are separable we restrict our considerations to one space dimension ($n = 1$). Making a Gaussian wave Ansatz for $\psi$ is equivalent to making the Ansätze

$$\theta_1(x, t) = -\frac{(x - s(t))^2}{2\sigma(t)^2} + \frac{1}{2} \ln(\sigma(t)) \tag{31}$$

$$\theta_2(x, t) = \iota_0 (A(t) x^2 + B(t) x + C(t)) . \tag{32}$$

Inserting these into the amplitude and phase equation (15) and equating equal powers of $x$ yields coupled nonlinear ordinary differential equations for $A, B, C, s$ and $\sigma$ as functions of $t$. This set of equations can be reduced to two ordinary second order equations in $\sigma$ and $s$

$$\ddot{\sigma} = \iota_3 \frac{(\dot{\sigma})^2}{\sigma} + 4(\iota_2\iota_3 + \iota_4)\frac{\dot{\sigma}}{\sigma^2} + 8\frac{(\iota_1 + \iota_5)}{\sigma^3} + 2\kappa\iota_0 \sigma \tag{33}$$

$$\ddot{s} = \left(\iota_3 \frac{\dot{\sigma}}{\sigma} + 2\frac{(\iota_2\iota_3 + \iota_4)}{\sigma^2}\right) \dot{s} + 2\kappa\iota_0 s \tag{34}$$

and $A, B,$ and $C$ may be expressed in terms of those. First one has to solve the decoupled equation (33) and then insert the result into (34) and to integrate that equation.

## 4.2 General Gaussian solutions

For the solution of equations (33) and (34) we will treat the free particle and the harmonic oscillator separately.

In case of the free particle ($\kappa = 0$) equation (33) can be reduced by the Ansatz

$$\frac{d}{dt}\sigma^2(t) = h\left(\ln(\sigma^2(t))\right) . \tag{35}$$

to the following first order differential equation for $h(z)$:

$$2h(z)h'(z) - (1+\iota_3)h(z)^2 - 8(\iota_2\iota_3 + \iota_4)h(z) - 32(\iota_1 + \iota_5) = 0 \tag{36}$$

Integrating this equation yields in general only an implicit equation for $h(z)$. In certain special cases we get an explicit expression, which can be substituted into the Ansatz (35). As an example let us treat the case $\iota_1 + \iota_5 = 0$, where

$$h(z) = \begin{cases} C_1 \exp\left(\frac{\iota_3+1}{2}z\right) - 8\frac{\iota_2\iota_3+\iota_4}{1+\iota_3} & \text{for } \iota_3 \neq -1 \\ 4(\iota_4 - \iota_2)z + C_1 & \text{for } \iota_3 = -1\,, \end{cases} \tag{37}$$

and we get the implicit solutions

$$\begin{aligned} \int_{\sigma_0}^{\sigma(t)} \frac{2x\,dx}{C_1 x^{\iota_3+1} - 8\frac{\iota_2\iota_3+\iota_4}{1+\iota_3}} &= t - t_0 \qquad \text{for } \iota_3 \neq -1\,, \\ \int_{\sigma_0}^{\sigma(t)} \frac{2x\,dx}{8(\iota_4-\iota_2)\ln(x) + C_1} &= t - t_0 \qquad \text{for } \iota_3 = -1\,. \end{aligned} \tag{38}$$

Explicit solutions are obtained for $\iota_3 = 1$

$$\sigma(t) = \sqrt{\frac{\exp(2\sigma_1 t - \sigma_1\sigma_2) + 2(\iota_2+\iota_4)}{\sigma_1}}\,, \tag{39}$$

or for $\iota_4 = -\iota_2\iota_3$, $\iota_3 \neq 1$

$$\sigma(t) = \sigma_0\left(\sigma_1 t + 1\right)^{\frac{1}{1-\iota_3}} \tag{40}$$

where $\sigma_1, \sigma_2$ are integration constants to be chosen suitably.

For the harmonic oscillator ($\kappa > 0$) we have not found an analogous Ansatz to reduce (33). Instead, for $\iota_3 \neq 1$ we transform (33) to an equation for the newly defined function

$$q(t) := \left(\sigma(t)\right)^{1-\iota_3} \tag{41}$$

such that the following two differential equations for $q, s$ describe the behaviour of the Ansatz (31) in (2)

$$\ddot{q} = 4(\iota_2\iota_3 + \iota_4)q^{\frac{2}{\iota_3-1}}\dot{q} - 8(\iota_1+\iota_5)(\iota_3-1)q^{\frac{\iota_3+3}{\iota_3-1}} - 2\kappa\iota_0(\iota_3-1)q\,, \tag{42}$$

$$\ddot{s} = \left(2(\iota_2\iota_3+\iota_4)q^{\frac{2}{\iota_3-1}} - \frac{\iota_3}{\iota_3-1}\frac{\dot{q}}{q}\right)\dot{s} + 2\kappa\iota_0 s\,. \tag{43}$$

The search for solutions of these equations is facilitated if we view them as one-dimensional Newtonian equations with friction. In this way we have the potentials

$$U_q(q) = \begin{cases} -32(\iota_1+\iota_5)q - 4\kappa\iota_0 q^2 & \text{if } \iota_3 = -3 \\ -16(\iota_1+\iota_5)\ln q - 2\kappa\iota_0 q^2 & \text{if } \iota_3 = -1 \\ 4\frac{(\iota_1+\iota_5)(\iota_3-1)^2}{\iota_3+1} q^{2\frac{\iota_3+1}{\iota_3-1}} + \kappa\iota_0(\iota_3-1)q^2 & \text{if } \iota_3 \neq -3,-1,1 \end{cases} \tag{44}$$

$$U_s(s) = \kappa\iota_0 s^2 \tag{45}$$

and the friction forces

$$F_q(q,\dot q) = 4(\iota_2\iota_3+\iota_4)q^{\frac{2}{\iota_3-1}}\dot q \tag{46}$$

$$F_s(\dot s) = \left(2(\iota_2\iota_3+\iota_4)q^{\frac{2}{\iota_3-1}} - \frac{\iota_3}{\iota_3-1}\frac{\dot q}{q}\right)\dot s\,. \tag{47}$$

Let us treat the case of the harmonic oscillator ($\kappa > 0$) first. The asymptotic motion of the $q$-particle depends on the potential $U_q$ and the friction force $F_q$. If $\iota_0 < 0 < \iota_1+\iota_5$, the potential has a minimum. Thus if furthermore $\iota_2\iota_3+\iota_4 < 0$ the friction force $F_q$ breaks the motion, and q tends to the minimum of the potential and so does $\sigma$:

$$\lim_{t\to\infty}\sigma(t) = \sigma_\infty := \left(\frac{4(\iota_1+\iota_5)}{-\kappa\iota_0}\right)^{1/4}, \qquad \lim_{t\to\infty}\dot\sigma(t) = 0\,. \tag{48}$$

As a consequence at some instant the friction force $F_s$ on the $s$ particle becomes negative and

$$\lim_{t\to\infty} s(t) = 0\,, \qquad \lim_{t\to\infty}\dot s(t) = 0\,. \tag{49}$$

The wave function $\psi(t)$ turns out to converge asymptotically to the ground state of the harmonic oscillator (25). Explicit solutions of (2) with this asymptotic behaviour are quasi-classical states with constant width $\sigma(t) = \sigma_\infty$ (see section 5.1).

If the above listed conditions are not fulfilled, then the width of the Gaussian wave packet might converge to infinity or to zero, i.e. to a constant or a $\delta$ distribution. Furthermore this behaviour can depend on the initial condition in the case $\iota_1+\iota_5 < 0 < \iota_0$ when the potential $U_q$ has a maximum and quasi-classical states with $\sigma(t) = \sigma_\infty$ may again be taken to illustrate the motion of the wave packet. While the ground state for $\iota_0 <$ is stable under small perturbations, it is unstable for $\iota_0 > 0$; a small perturbation leads to a de- or increasing width and a motion of the centre $s(t)$ away from the origin.

For another discussion of the stability of solutions of (1) see [6] in this volume.

# 5 Solitary wave solutions

## 5.1 Gaussian solitary waves

Gaussian solitary waves are Gaussian waves whose width remains constant: $\sigma(t) = \sigma_0 = \mathit{const}$. Requiring this to be the case reduces (33) to

$$\kappa\iota_0(\sigma_0)^4 + 4(\iota_1+\iota_5) = 0\,. \tag{50}$$

In the free case ($\kappa = 0$) this can only be satisfied if $\iota_1 + \iota_5 = 0$, for the harmonic oscillator $\sigma_0$ has to be the width $\sigma_\infty$ (48) of the ground state. Equation (34) reduces to an ordinary differential equation of second order:

$$\ddot{s} = 2(\iota_2\iota_3 + \iota_4)\sigma_0^{-2}\dot{s} + 2\kappa\iota_0 s \tag{51}$$

For $\kappa > 0$ and $\iota_0 < 0$ ($\iota_0 > 0$) this is the equation of motion of a damped harmonic (anti-)oscillator, for $\kappa = 0$ the equation of motion of a damped free particle. Again the gauge invariant expression $\iota_2\iota_3 + \iota_4$ determines whether we have damping, pumping or no friction at all.

### 5.2 Other solitary waves

Suppose now $\theta = (\theta_1(\vec{x}), \theta_2(\vec{x}, t))$ is a solution of the free ($V \equiv 0$) equations(20) and (21) as in section 3.2 and that in addition we have no friction forces for the Gaussian solutions, $\iota_2\iota_3 + \iota_4 = 0$, then it turns out that

$$\tilde{\theta}_1(\vec{x}, t) = \theta_1(\vec{x} - \vec{v}t)\,, \tag{52}$$

$$\tilde{\theta}_2(\vec{x}, t) = \theta_2(\vec{x} - \vec{v}t, t) - \frac{1}{2}\vec{v}\cdot\vec{x} + \frac{v^2}{4}(1 - \iota_3)t \tag{53}$$

is a *non-stationary solution of the amplitude and phase equation (15) and (16)*. It should be emphasized that the original equation (2) is only Galilei invariant if $\iota_3 = 0 = \iota_4$ but that the transformations (52, 53) take solutions to solutions in the more general case $\iota_2\iota_3 + \iota_4 = 0$. The resulting solution of (2) is then

$$\psi(\vec{x}, t) = \psi_0 \exp\left(\theta_1(\vec{x} - \vec{v}t) + i(\iota_2\theta_1(\vec{x} - \vec{v}t) - \frac{1}{2}\vec{v}\cdot\vec{x} + \frac{v^2}{4}(1 - \iota_3)t - \omega t)\right)\,, \tag{54}$$

where $\theta_1$ is a solution of (21) and $\psi_0 = const.$ Consequently the probability density $\rho(\vec{x}, t) = \psi(\vec{x}, t)\bar{\psi}(\vec{x}, t)$ becomes a solitary wave

$$\rho(\vec{x}, t) = |\psi_0|^2 \exp(2\theta_1(\vec{x} - \vec{v}t)) \tag{55}$$

moving with constant speed without changing shape.

In particular for $\iota_1(\iota_1 + \iota_5) < 0$ we get square integrable solutions of the type

$$\begin{aligned}\psi(\vec{x}, t) = {} & \psi_0 \cosh\left(\vec{k}\cdot(\vec{x} - \vec{v}t)\right)^{\frac{\iota_1}{\iota_1+\iota_5}} \exp\left\{ i\frac{2\iota_2\iota_1}{\iota_1 + \iota_5} \ln\left[\cosh\left(\vec{k}\cdot(\vec{x} - \vec{v}t)\right)\right]\right. \\ & \left. -2\frac{\iota_1^2}{\iota_1 + \iota_5}k^2 t\right\}\end{aligned} \tag{56}$$

### 5.3 Solitary waves with arbitrary initial data

If the parameters are as in the previous section, i.e. $\iota_2\iota_3 + \iota_4 = 0$ and in addition $\iota_1 = 0 = \iota_5$ then (as was shown in section 3.2.2) the phase equation (21) is satisfied by *arbitrary* functions $\theta_1(\vec{x})$. Hence, it follows that for the three parameter family selected by the conditions

$$\iota_1 = \iota_2\iota_3 + \iota_4 = \iota_5 = 0 \tag{57}$$

the nonlinear equation (2) has solutions

$$\psi(\vec{x},t) = \exp\left(\theta_1(\vec{x}-\vec{v}t) + i(\iota_2\theta_1(\vec{x}-\vec{v}t) - \frac{1}{2}\vec{v}\cdot\vec{x} + \frac{v^2}{4}(1+\frac{\iota_4}{\iota_2})t - \omega t)\right) \tag{58}$$

with *arbitrary* $\theta_1(\vec{x})$. Just as in the previous section this leads to solitary wave motion for the "probability density" $\rho(\vec{x},t)$ but now with arbitrary initial data.

## 6 Conclusion

We have presented numerous explicit solutions for members of the family of nonlinear equations (2). These solutions are constructed for the parameters defined by $\nu_1 = 1, \mu_1 = 0$ and contain the parameter combinations which are invariant under the nonlinear transformation (10). Using this transformation and the diagram in (11) solutions for all other values of $\nu_1$ and $\mu_1$ are easily obtained.

The solutions given here contain plane waves, nontrivial stationary solutions, stationary solutions associated with a linear Schrödinger equation, Gaussian wave packets, and various solitary waves including a three parameter subfamily of (2) which permits solitary waves for arbitrary initial data. Recently R. Zhdanov has found other explicit solutions of (2) for the case $\iota_3 \neq 0 \neq \iota_5$, $\iota_1 = \iota_2 = \iota_4 = 0$ [21].

It seems quite remarkable that despite its nonlinear character so many explicit solutions of equation (2) can be found. Apart from its possible use in other fields than quantum mechanics the knowledge of ever more explicit solutions may help to shed light on the physical meaning (if any) of the various parameters.

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## References

[1] H.-D. Doebner and G.A. Goldin, "On a general nonlinear Schrödinger equation admitting diffusion currents", Phys. Lett. **162A**, 397 (1992).

[2] H.-D. Doebner and G.A. Goldin, "Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations", J. Phys. **A27**, 1771 (1994).

[3] G.A. Goldin, "Diffeomorphism group representations and quantum nonlinearity: gauge transformations and measurement", this volume.

[4] J.D. Hennig, "Nonlinear Schrödinger equations and Hamiltonian mechanics", this volume.

[5] W. Lücke, "Nonlinear Schrödinger dynamics and nonlinear observables", this volume.

[6] G.A. Goldin, B.A. Malomed, and L. Stenflo, "Modulational stability of plane waves for a class of nonlinear Schrödinger equations", this volume.

[7] S.S. Mizrahi and V.V. Dodonov, "Gaussian solutions of Doebner-Goldin and Kostin nonlinear equations", this volume.

[8] T.W.B. Kibble, "Relativistic models of nonlinear quantum mechanics", Comm. Math. Phys. **64**, 73–82 (1978).

[9] F. Guerra and M. Pusterla, "A nonlinear Schrödinger equation and its relativistic generalization from basic priciples", Lett. Nuovo Cim. **34**, 351–356 (1982).

[10] L. Smolin: "Quantum fluctuations and inertia", Phys. Lett. A **113**, 408–412 (1986).

[11] J.-P. Vigier, "Paricular solutions of a non-linear Schrödinger equation carrying particle-like singularities represent possible models of de Broglie's double solution theory", Phys. Lett. A **135**, 99–105 (1989).

[12] P.C. Sabatier, "Multidimensional nonlinear Schrödinger equations with exponentially confined solutions", Inverse Problems **6**, L47–L53 (1990).

[13] B.A. Malomed and L. Stenflo: "Modulational instabilities and soliton solutions of a generalized nonlinear Schrödinger equation", J. Phys. A: Math. Gen. **24**, L1149–L1153 (1991).

[14] Orfeu Bertolami: "Nonlinear corrections to quantum mechanics from quantum gravity", Phys. Lett. A **154**, 225–229 (1991).

[15] P. Nattermann, "Solutions of the general Doebner-Goldin equation via nonlinear transformations", in: "Proceedings of the XXVI Symposium on Mathematical Physics, Torun, December 7–10, 1993", p. 47, Nicolas Copernicus University Press, Torun (1994).

[16] G. Auberson and P.C. Sabatier, "On a class of homogeneous nonlinear Schrödinger equations", J. Math. Phys. **35**, 4028 (1994).

[17] H.-D. Doebner, G.A. Goldin, and P. Nattermann, "A family of nonlinear Schrödinger equations: linearizing transformations and resulting structure". Clausthal-preprint ASI-TPA/8/94, to appear in the proceedings of the XIIIth Workshop on Mathematical Methods in Physics, Bialowieza, July 9–15 1994, 1995.

[18] P. Nattermann, W. Scherer, and A.G. Ushveridze, "Exact solutions of the general Doebner-Goldin equation". Phys. Lett. **184A**, 234 (1994).

[19] P. Nattermann, W. Scherer, and A.G. Ushveridze, "Correlated coherent states of the general Doebner-Goldin equation". to appear in Int. J. Mod. Phys. B, 1995.

[20] V.V. Dodonov and S.S. Mizrahi, "Doebner-Goldin nonlinear model of quantum mechanics for a damped oscillator in a magnetic field". Phys. Lett. **181A**, 129 (1993).

[21] R. Zhdanov, private communication.